On the efficient maintenance and updating of Pareto solutions when assigned objectives values may change

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Abstract

Usually when undertaking a multi-objective optimisation problem it is assumed that on evaluation of a design, the assigned objectives are fixed. This allows for domination comparisons to be undertaken at just one time step to decide on whether a design should be categorised as an elite (non-dominated) solution, with the designation only removed if a new location is found to be better at a later time step. However, there are some situations where the objective vector assigned to a design may change at a later time point. This may be due to some global change in the environment (a dynamic problem), which effects all designs proposed thus far, however it can also be for a single solution in isolation. This may for instance be due to resampling in a noisy domain causing an update in the *estimated* objective values associated with a solution, or via an increased resolution of the objective evaluation (e.g. a finer mesh on a finite element analysis of a design, or more data instances for a classifier being tuned).

How to efficiently maintain an elite archive when the assigned objectives are susceptible to change has not been widely addressed in the literature thus far. Although a number of data structures exist for efficiently maintaining and querying solutions, they assume that a domination relationship between two designs at time t, will persist at all future time steps. When this no longer holds, if we still want to guarantee access to the best estimate of the non-dominated subset of solutions visited by an optimiser at any time step, it becomes necessary to track dominated as well as non-dominated solutions as the search progresses,.

Here we discuss different storage and query approaches which guarantee this, and propose a novel data maintenance regime based on chaining single domination links between all solutions evaluated at any time point to rapidly discern the non-dominated subset both as existing solutions have their assigned objective values changed, and as new design locations are proposed. We detail the computational complexity of this approach, and compare the empirical performance of three different link selection protocols on simulated behaviour of set updates, mimicking a converging optimiser and a optimiser performing a random search, and where locations are resampled at random, or resampling based on their estimated non-dominance. We also present results when optimising the ten problems from the CEC'09 test suite.

1 Introduction

The typical assumption when performing an optimisation, uni-objective or multi-objective, is that the evaluated objective(s) for a solution do not vary, unless the design itself is modified. This is not the case for all problem types however. In dynamic optimisation problems [1] the problem *itself* changes over time, meaning the objective vectors of all solutions evaluated at any time point may vary if reassessed at a later time point. Also in some noisy domains repeated evaluation of the same design will lead to different objective evaluations, if the noise experienced modifies the objective evaluation of a solution, or modifies the design prior to the evaluation by the objective function [2, 3, 4]. Finally, the objective evaluation may be updated/refined at some later point due to increased fidelity of the model assessment (e.g. finite element, or data dependent evaluation functions [5]).

In all these cases, the objective vector $\mathbf{y}$ associated with the design vector $\mathbf{x}$ at some time t, may not be the same as the objective vector at $t+1$.

We may consider the problem in a general form by examining a change to the objectives $\mathbf{y}$ associated with single $\mathbf{x}$, out of the set of all designs evaluated by time t by an optimiser. We represent this set as X^t, and the non-dominated subset as A^t, with their mapped objective space locations being Y^t and E^t respectively. In a situation where n *multiple* members of X are changed, each individual change may be viewed as a distinct time step for our purposes (the order these are processed being immaterial to the final state of A^{t+n}).

If the $\mathbf{y}$ associated with $\mathbf{x}$ varies between t and $t+1$, and $\mathbf{x} \in A^t$, $\mathbf{x}$ may no longer be in A^{t+1} due to the shift in its $\mathbf{y}$. As such solutions marked as dominated at time t may now be evaluated as non-dominated at $t+1$, and enter A^{t+1}.

We now briefly review the ideas of dominance and Pareto optimality before discussing the computational complexity of managing a set where at each time step a previously evaluated solution may have its objective vector changed, or a brand new solution may be evaluated, and introduce a data structure and management algorithm to mitigate the cost of maintaining A^t and E^t.

2 Multi-objective optimisation

A general multi-objective optimisation problem seeks to simultaneously extremise D objectives:

$$f_d(\mathbf{x}), d = 1, \ldots, D \tag{1}$$

where each objective depends upon a vector

$$\mathbf{x} = (x_1, \ldots, x_l, \ldots, x_L) \tag{2}$$

of K parameters or design variables.

The parameters may also be subject to equality and inequality constraints which, for simplicity, we assume can be evaluated precisely. When the objectives are to be minimised, the multi-objective optimisation problem may thus be expressed as:

$$\text{minimise} \qquad \mathbf{y} = \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_D(\mathbf{x})) \tag{3}$$

with $\mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^L$, the feasible domain as defined by any design constraints. When faced with only a single objective an optimal solution is one which minimises the objective given the model constraints. However, when there is more than one objective to be minimised, solutions may exist for which performance on one objective cannot be improved without reducing performance on at least one other. Such solutions are said to be *Pareto optimal*.

The notion of *dominance* may be used to make Pareto optimality clearer. Assuming, without loss of generality, that the goal is to minimise the objectives, a design vector $\mathbf{x}$ is

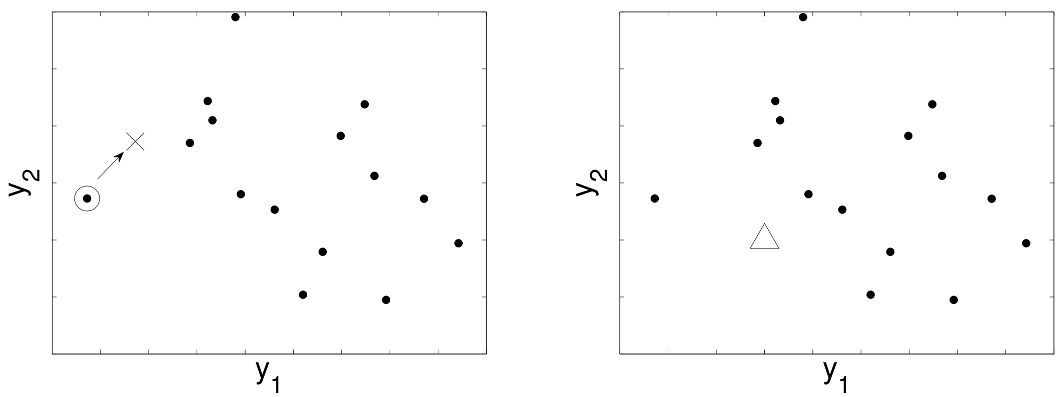

Figure 1: Illustration of the two possible transitions from Y^t to Y^{t+1}. *Left:* a single member of X^t has its $\mathbf{y}$ shifted (indicated with a circle and a cross). *Right:* a new design location has been evaluated, indicated with a triangle.

said to *strictly dominate* another $\mathbf{u}$ iff

$$f_d(\mathbf{x}) \leq f_d(\mathbf{u}) \quad \forall d = 1, \ldots, D \quad \text{and} \quad \mathbf{f}(\mathbf{x}) \neq \mathbf{f}(\mathbf{u}). \tag{4}$$

This is often denoted as $\mathbf{x} \prec \mathbf{u}$ (as opposed to $\mathbf{f}(\mathbf{x}) \prec \mathbf{f}(\mathbf{u})$). A set of design vectors A is said to be a *mutually non-dominating set* if no member of the set is dominated by any other member. A solution to the minimisation problem (3) is thus *Pareto optimal* if it is not dominated by any other feasible solution, and the non-dominated set of all Pareto optimal solutions is the Pareto set $\mathcal{P}$.

In this paper we are concerned with problems in which the objectives associated with a solution may *vary* from one time step to the next. As such we effectively have an objective vector $\mathbf{y}$ which is dependent on t, and therefore indicate this with $\mathbf{y}^t$. The dominance comparison $\mathbf{x} \prec \mathbf{u}$ is therefore dependent on the time step t at which this comparison takes place, as $\mathbf{x} \prec \mathbf{u}$ requires access to the objective vectors associated with $\mathbf{x}$ and $\mathbf{u}$.

An illustration is provided in Figure 1. In the left panel a single member of X^t, $\mathbf{x}_j$ has had its objective vector in Y^t shifted. (Note that for all other $\mathbf{x}_i \in X^t$, $i \neq j$, $\mathbf{y}_i^t = \mathbf{y}_i^{t+1}$, and $|Y^t| = |Y^{t+1}|$.) In the right panel a new member has entered X^{t+1}, meaning $|Y^{t+1}| = |Y^t| + 1$.

As we are exclusively concerned with the objective space representation of solutions, we shall restrict ourselves to Y^t and E^t for the rest of this paper, and denote the dominance relationship by directly comparing solutions in objective space, i.e. if ith objective vector in Y^t dominates the jth objective vector in Y^t, then $\mathbf{y}_i^t \prec \mathbf{y}_j^t$.

3 Best case computational complexity of updating on a single time step

Here we consider a general set Y^t where t denotes the state of all known objective vectors at a particular time. As t increases, either a single solution has had its associated objective vector in Y *modified*, or a new location, has been evaluated for the first time, resulting in $\mathbf{y}^{t+1}_{new}$, which is added to Y. In the first instance, if the jth member of Y is changed, then $\mathbf{y}^t_j \neq \mathbf{y}^{t+1}_j$, however for all other elements in Y^t, $\mathbf{y}^t_i = \mathbf{y}^{t+1}_i$. In the second situation for all solutions in Y^t, $\mathbf{y}^t_i = \mathbf{y}^{t+1}_i$, however Y^{t+1} has one more member than Y^t.

The non dominated members E^t of Y^t may be selected via the `nondom` function defined below:

$$E^t = \texttt{nondom}(Y^t) = \{\mathbf{y}^t \in Y^t | \nexists \mathbf{v}^t \in Y^t, \mathbf{v}^t \prec \mathbf{y}^t\}. \tag{5}$$

Alternatively, the members of Y^t which dominate any particular element of the set, $\mathbf{v}^t$, may be determined via the `dom_members` function:

$$V^t = \texttt{dom\_members}(\mathbf{v}^t, Y^t) = \{\mathbf{y}_t \in Y^t \setminus \mathbf{v}^t | \mathbf{y}^t \prec \mathbf{v}^t\}. \tag{6}$$

If $\texttt{dom\_members}(\mathbf{v}^t, Y^t) = \emptyset$, then $\{\mathbf{v}^t\} \subseteq E^t$. Conversely if $\texttt{dom\_members}(\mathbf{v}^t, Y^t) \neq \emptyset$, then $\mathbf{v}^t \notin E^t$. E^t may thus be alternatively defined via the `dom_members` function:

$$E^t = \{\mathbf{y}^t \in Y^t | \texttt{dom\_members}(\mathbf{y}^t, Y^t) = \emptyset\}. \tag{7}$$

This may seem a slightly convoluted route to defining the non-dominated set, however it is a useful formulation when considering how the effect of varying the objective vector associated with a design location may be used when deciding which solutions need to be compared to E^t for possible entry into E^{t+1}

3.1 Shifting the objective vector of a dominated member of Y^t

Consider the effect of changing the objective vector of the jth member of Y^t.

If $\mathbf{y}_j \notin E^t$, then the change to $\mathbf{y}^{t+1}_j$ (from $\mathbf{y}^t$) may mean it should now enter E^{t+1}, if $\texttt{dom\_members}(\mathbf{y}^{t+1}, E^t) = \emptyset$. However, there are no other members of Y^{t+1} that can enter E^{t+1} as a result of the movement of $\mathbf{y}^t_j$ to $\mathbf{y}^{t+1}_j$, as if they were dominated by members of Y at time t, they will still be dominated by the same members at $t+1$ (i.e. as $\mathbf{y}^t_j \notin E^t$, then there must be at least one $\mathbf{e}^t_k \in E^t$ which dominates $\mathbf{y}^t_j$. This $\mathbf{e}^t_k$ therefore also dominates *all* solutions that $\mathbf{y}_j$ dominated at time t at time $t+1$.)

Existing members of E^t, $\mathbf{e}^t_i$, may however require removal from E^{t+1} if they are dominated by the new $\mathbf{y}^{t+1}_j$ location, so if $\mathbf{y}^{t+1}_j$ enters E^{t+1}, then each $\mathbf{e}^t_i \in E^t$ will need to be compared to $\mathbf{y}^{t+1}_j$, and if $\mathbf{y}^{t+1}_j \prec \mathbf{e}^t_i$ then $\mathbf{e}^t_i$ cannot be a member of E^{t+1}. As long as the domination comparison results are stored when $\mathbf{y}^{t+1}_j$ is compared to E^t for entry into E^{t+1}, this stage will not require any additional domination calculations, meaning a maximum of $|E^t|$ domination comparisons are required when shifting the objective vector of a dominated member of Y^t.

3.2 Varying a non-dominated member of Y^t

If prior to varying its objectives $\{\mathbf{y}_j^t\} \subseteq E^t$, then its location shift may mean it should not be in E^{t+1}. However, *even* if it does enter E^{t+1}, solutions that it dominated in Y^t may need to enter E^{t+1} as the shift in location to $\mathbf{y}_j^{t+1}$ from $\mathbf{y}_j^t$ may mean solutions in Y^t previously dominated by $\mathbf{y}_j^t$ are no longer dominated in Y^{t+1}.

If $\texttt{dom\_members}(\mathbf{y}_j^{t+1}, E^t \setminus \{\mathbf{y}_j^t\}) = \emptyset$, then $\mathbf{y}_j^{t+1}$ will enter E^{t+1}, otherwise it will be excluded and E^{t+1} is initially set as E^t. If $\mathbf{y}_j^{t+1}$ does enter E^{t+1}, then all other $\mathbf{e}_i^t \in E^t$ will need to be assessed to see if they are dominated. If $\mathbf{y}_j^{t+1} \prec \mathbf{e}_i^t$, then $\mathbf{e}_i^t$ will not be a member of E^{t+1}.

The members $\mathbf{y}_i^t$ of Y^t which are dominated by $\mathbf{y}_j^t$ need to be reevaluated with respect to $\mathbf{y}_j^{t+1}$ (for potential entry into E^{t+1}) if $\texttt{dom\_members}(\mathbf{y}_i^t, Y^t) = \{\mathbf{y}_j^t\}$ – that is, they are *only* dominated by $\mathbf{y}_j$ at time t. If a $\mathbf{y}_i^t$ is dominated by additional members of Y^t, then it will still be dominated by these same members of Y^{t+1}, as their objective vector values will be unchanged.

We denote by K the number of solutions in Y^t which are *exclusively* dominated by $\mathbf{y}_j^t$. In practise K is typically very small (often zero) – as to be *only* dominated by a single member of Y^t a solution must lie in the level 2 Pareto shell [6] (those that would be non-dominated where the E^t members of Y^t not to exist), and be in close proximity to the dominating member.

In the worst case there are $|E^t| + K - 1$ domination comparisons required before the complete membership of E^{t+1} is determined. That is, $\mathbf{y}_j^{t+1}$ will at worst need comparing to all archive members at time t bar itself ($|E^t| - 1$ comparisons) and the K solutions dominated only by $\mathbf{y}_j^t$ will need comparing to $\mathbf{y}_j^{t+1}$ (resulting in K domination comparisons). If they are not dominated by $\mathbf{y}_j^{t+1}$ they may enter $E^t + 1$ directly without any other domination comparisons required, as if $\mathbf{y}_j^t$ was the *only* dominating member in Y^t, then by definition no other members of E^t can dominate them.

3.3 Sampling a new location

Finally, a completely new location, $\mathbf{y}_{new}^{t+1}$, may be suggested. In this case we may determine membership or exclusion from E^{t+1} by domination comparison of this proposal solely with the members of E^t.

From this we can see the computational complexity of determining the elite set of solutions when adding a new location into Y^{t+1} is in the worst case $|E^t|$ domination comparisons.

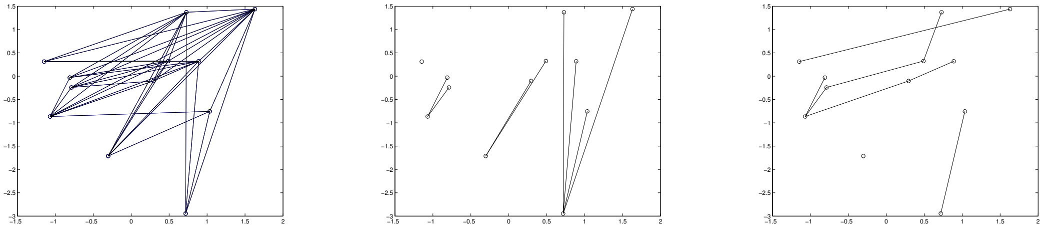

Figure 2: Graph representations of a general set of 12 solutions, plotted via their objective values in Y^t. *Left:* All domination relations between solutions shown via edges. *Middle:* Single domination relation edge plotted per solution, only members of E^t selected as dominating member (plotted via their objective values in E^t). *Right:* Single domination relation edge shown per solution, dominating member selected at random from valid candidates in Y^t.

4 Implications for multiple sequential updates

On inspection of the properties above, the cost of determining the non-dominated membership of Y^{t+1} (i.e. the membership of E^{t+1}), when a single member is varied is the same as a standard update due to a new solution proposal when $\mathbf{y}_j^t$ is dominated, and has additional K comparisons required when $\mathbf{y}_j^t$ is non-dominated – assuming that all the domination links between Y^t are stored somewhere. Additionally, K tends to be zero or very small due to the geometric properties of a set of points when using domination comparisons, as points which are only dominated by a single member of Y^t must lie in a region of objective space which is exclusively dominated by that single element of Y^t (i.e. outside of the volumes of objective space dominated by *all* other Y^t members).

However, if *only* these domination comparisons are made when moving from Y^t to Y^{t+1}, then updates to the domination relationships between general members of Y^{t+1} will not be recorded, which are needed for exploitation in subsequent time steps using the approach outlined above to identify the membership of E.

For instance, in order to exploit the properties discussed in section 3.3, which rely on a solution knowing *which* members of Y^t it is dominated by, we will also need to compare a new location $\mathbf{y}_{new}^{t+1}$ against $Y^t \setminus E^t$ to keep track of this information. Specifically, if $\mathbf{y}_{new}^{t+1}$ enters E^{t+1}, then the members of Y^t it dominates will need to be identified (a worst case $|Y^{t+1} \setminus E^{t+1}|$ domination comparison operation).

If $\mathbf{y}_{new}^{t+1}$ is dominated by E_t, then we can utilise the observation that $\mathbf{y}_{new}^{t+1}$ can at most dominate the subset of Y^{t+1} which those elements of E^{t+1} who dominate $\mathbf{y}_{new}^{t+1}$ dominate collectively, but in the worst case this too is also a $|Y^{t+1} \setminus E^{t+1}|$ operation.

The maintenance of all domination links between members of Y^t as new solutions enter it is therefore exorbitant for any $|Y^t|$ larger than a modest size. Likewise, the variation of the $\mathbf{y}_j^t$ for an existing member of Y^t may also require a comparison of $\mathbf{y}_j^{t+1}$ to the entirety of Y^t should $\mathbf{y}_j^{t+1}$ enter E^{t+1}.

Given this overhead, it is impractical to use the approach outlined in section 3 directly, however, we can obtain empirical update properties approaching those detailed by maintaining the information for just a *single* dominating solution per member of $Y^t \setminus E^t$. This has vastly reduced storage requirements, and the computational cost of maintaining these single links is also far superior to maintaining *all* domination links. We will now outline this proposed approach.

5 Exploiting single domination tracking

Instead of keeping track of *all* domination relationships between members of Y^t, here we suggest that for each member $\mathbf{y}_i^t$ of Y^t a *single* other solution in Y^t is tracked which dominates it, which we label $\mathbf{y}_i^{t,*}$ for convenience. If there is no solution in Y^t which dominates a particular $\mathbf{y}_i^t$, then $\{\mathbf{y}_i^t\} \subseteq E^t$. Example plots are provided in Figure 2 which illustrate the properties of maintaining these single relationships. In the left panel of Figure 2, 12 solutions are plotted in two dimensional objective space. The 47 domination relationships between these solutions are plotted via edges between each solution. The middle panel plots the same set, but with only a single domination relationship plotted linking $\mathbf{y}_j^t$, where all $\mathbf{y}_i^{t,*}$ which dominate members of $Y^t \setminus E^t$ are members of E^t (making 8 edges in total as $|Y^t \setminus E^t| = 8$). The right panel plots Y^t, with the $\mathbf{y}_i^{t,*}$ which dominate each $Y^t \setminus E^t$ element selected arbitrarily from the set of potential candidates (again resulting in 8 edges).

A number of properties can be observed with the aid of the illustration provided in Figure 2. The first is the substantial number of domination relations that can exist in a general set of data between its elements; the minimum is $|Y^t \setminus E^t|$, which corresponds to each element of $Y^t \setminus E^t$ being dominated by a single element of E^t – i.e. there are two Pareto shells, with each element of E^t dominating non intersecting subsets of the second shell. The maximum follows a triangular number sequence: $(|Y^t| - 1)(|Y^t| - 2)/2$, which corresponds to the situation where the number of Pareto shells equals the number of elements in Y^t, meaning the first shell element dominates $|Y^t| - 1$ solutions, the next shell element dominates $|X^t| - 2$ solutions, and so on. The number of links maintained when only tracking a single dominating link per element is equivalent to the smallest number of domination links possible for a set Y^t which has $|E^t|$ non-dominated members.

On inspection of the right panel of Figure 2, it can be seen that if the dominating solution of an element $\mathbf{y}_i^t$, $\mathbf{y}_i^{t,*}$, is itself dominated, then there is a chain of domination links which eventually reaches a member of E^t. This is inevitable, as all solutions have a dominating member associated with them, bar the members of E^t themselves. As such all chains end with an E^t solution. Chains may join together as they approach E^t, but it is impossible for a dominated solution to not have a direct sequence to E^t via the maintained dominating member links. This property is extremely useful, and we exploit it in our proposed approach for maintaining Y^t and tracking E^t. We denote the subset of Y^t which has $\mathbf{y}_j^t$ tracked as its single dominator by $Y^t_{\mathbf{y}_j^t}$ – note that $\mathbf{y}_j^t$ may not be the only element in Y^t which dominates the members of $Y^t_{\mathbf{y}_j^t}$, it is merely the single dominator that is being actively tracked.

5.1 Changing the objective vector of a *dominated* member of X^t

Consider the situation when a single dominated member of Y^t, $\mathbf{y}_j^t$, has its location changed, when a single dominating solution is tracked per element.

- If prior to alteration $\mathbf{y}_j^t \notin E^t$, then its change in location may mean $\mathbf{y}_j^t$ should now enter E^{t+1}. The first domination check is against $\mathbf{y}_j^{t,*}$. If $\mathbf{y}_j^{t,*} \prec \mathbf{y}_j^{t+1}$, then it need not be compared to any other solution and $\mathbf{y}_j^{t,*}$ remains the tracked single dominator (although the time step increment will mean it is now labelled as $\mathbf{y}_j^{t+1,*}$).
- If $\mathbf{y}_j^{t+1}$ is no longer dominated by $\mathbf{y}_j^{t,*}$, it is then compared against the members of E^t (note, $\mathbf{y}_j^{t,*}$ may be a member of E^t). If $\texttt{dom\_members}(\mathbf{y}_j^{t+1}, E^t) \neq \emptyset$ then $E^{t+1} = E^t$, and $\mathbf{y}_j^{t+1,*}$ is selected as one of the subset of E^{t+1} which dominates $\mathbf{y}_j^{t+1}$.
- If $\mathbf{y}_j^{t+1}$ is not dominated by any member of E^t, then it enters E^{t+1}, and any elements of E^t which are not dominated by $\mathbf{y}_j^{t+1}$ are also added to E^{t+1} (and there is no $\mathbf{y}_j^{t+1,*}$ stored). Those $\mathbf{e}_i^t \in E^t$ which are dominated by $\mathbf{y}_j^{t+1}$, and therefore do not enter E^{t+1}, have $\mathbf{y}_j^{t+1}$ set as their respective $\mathbf{e}_i^{t+1,*}$.
- As $\mathbf{y}_j^t$ has changed location, then the members of $Y^t_{\mathbf{y}_j^t}$ may no longer be dominated by $\mathbf{y}_j^{t+1}$. They are first checked against $\mathbf{y}_j^{t+1}$, if they are dominated, this becomes their $\mathbf{y}_i^{t+1,*}$, otherwise $\mathbf{y}_j^{t,*}$ is assigned this role – as this member dominated $\mathbf{y}_j^t$, and therefore must dominate all solutions which had $\mathbf{y}_j^{t+1}$ as their single tracked dominator at time t.

There are therefore two distinct numbers of domination comparisons that can be required when a dominated member of X^t has a $\mathbf{y}_j^t$ changed:

1. If $\mathbf{y}_j^{t+1}$ is still dominated by $\mathbf{y}_j^{t,*}$, then 1 domination comparison is required before the membership of E^{t+1} is determined
2. If $\mathbf{y}_j^{t+1}$ is not dominated by $\mathbf{y}_j^{t,*}$, then there are a maximum of $|E^t \cup \mathbf{y}_j^{t,*}|$ domination comparisons required before the membership of E^{t+1} can be determined.

In both cases a further $|Y^t_{\mathbf{y}_j^t}|$ domination comparisons are required to maintain the single domination links in Y^{t+1}.

5.2 Changing the objective vector of a *non-dominated* member of Y^t

Consider the altering of a non-dominated member of Y^t, $\mathbf{y}_j^t$, when maintaining a single dominating solution per element of $Y^t \setminus E^t$. In this case there is no $\mathbf{y}_j^{t,*}$ associated with the $\mathbf{y}_j^t$, as $\mathbf{y}_j^t \in E^t$.

- To determine if $\mathbf{y}_j^{t+1}$ is still non-dominated, it is first compared against the members of E^t. If $\texttt{dom\_members}(\mathbf{y}_j^{t+1}, E^t \setminus \{\mathbf{y}_j^t\}) \neq \emptyset$ then E^{t+1} is initially set as $E^t \setminus \{\mathbf{y}_j^t\}$.

- If $\mathbf{y}_j^{t+1}$ is not dominated by any element of E^t then it enters E^{t+1} and any elements of E^t which are not dominated by $\mathbf{y}_j^{t+1}$ are also added to E^{t+1}. Those $\mathbf{e}_i^t \in E^t$ which are dominated by $\mathbf{y}_j^{t+1}$, and therefore do not enter E^{t+1}, have $\mathbf{y}_j^{t+1}$ set as their respective $\mathbf{e}_i^{t,*}$.

- Those $\mathbf{y}_i^t \in Y^t \setminus E^t$ which have $\mathbf{y}_j^t$ as their $\mathbf{y}_i^{t,*}$ are first compared to $\mathbf{y}_j^{t+1}$ to see if they are dominated by its new location. If not, they also need to be compared to E^t to see if they should now be members of E^{t+1} due to the shift in location of $\mathbf{y}_j^{t+1}$ (as, although a single dominator is tracked, this may not be the *only* dominating member in Y^t of a $\mathbf{y}_i^t$). If any element of $Y_{\mathbf{y}_j^t}^t$ is not dominated by any members of $E^{t+1} \cup Y_{\mathbf{y}_j^t}^t$, then they are added to E^{t+1}. For each member of $Y_{\mathbf{y}_j^t}^t$ which is dominated by members of $E^{t+1} \cup Y_{\mathbf{y}_j^t}^t$, their $\mathbf{y}_i^{t,*}$ is selected as one of these dominating members. (We compare the empirical effect of different selection protocols for this in section 5.4.)

Therefore in the worst case there are $|E^t|(1 + |Y_{\mathbf{y}_j^t}^t|^2) - 1$ domination comparisons required when the objective values of a single member of E^t are changed. In practice, as we shall see later, $Y_{\mathbf{y}_j^t}^t$ is often empty, or with only a very few elements.

5.3 Sampling a new location

Finally, a completely new location, $\mathbf{y}_{new}^{t+1}$, may be suggested. In this case we may determine membership or exclusion from E^{t+1} by domination comparison of this proposal solely with the members of E^t. If it is dominated, then $\mathbf{y}_{new}^{t+1,*}$ is selected as a dominating member from E^{t+1}. If it dominates any $\mathbf{e}_i \in E^t$, then the corresponding $\mathbf{e}_i^{t,*}$ are set as $\mathbf{y}_{new}^{t+1}$.

From this we can see the worst case complexity when sampling a new location for entry into Y^{t+1} is $|E^t|$ domination comparisons.

5.4 Empirical comparison of protocols for selecting a single tracked dominating solution

As detailed in section 5.2, when reassigning $\mathbf{y}_i^{t+1,*}$ a new member is selected in the situation where:

1. $\mathbf{y}_i^{t,*}$ has been moved to a position which no longer dominates $\mathbf{y}_i^{t+1}$, or
2. $\mathbf{y}_i^t$ itself has moved and is no longer dominated by $\mathbf{y}_i^{t,*}$.

otherwise $\mathbf{y}_i^{t+1,*} = \mathbf{y}_i^{t,*}$.

In case 1 if $\mathbf{y}_j^t \notin E^t$ then the $\mathbf{y}_i^{t+1,*}$ are simply set to $\mathbf{y}_j^{t,*}$, otherwise $\mathbf{y}_i^{t+1,*}$ is selected from $V_t = \texttt{dom\_members}(\mathbf{y}_i^t, E^{t+1} \cup Y_{\mathbf{y}_j^t}^t)$. In case 2 $\mathbf{y}_i^{t+1,*}$ is selected from $V_t = \texttt{dom\_members}(\mathbf{y}_i^t, E^{t+1})$.

Here we examine the empirical complexity of three different methods for selecting this dominating solution from V^t (as opposed to the worst case complexities derived above). The three protocols we examine are:

1. The selected $\mathbf{y}_i^{t+1,*}$ is the closest (Euclidean) dominating member in V_t (distance being measures in objective space based on their estimated objective values).
2. The selected $\mathbf{y}_i^{t+1,*}$ is chosen at random from $\texttt{dom\_members}(\mathbf{y}_i^t, E^{t+1})$ only.
3. The selected $\mathbf{y}_i^{t+1,*}$ is chosen at random from the dominating members of V_t.

By selecting the closest member, it is guaranteed that if multiple members of $E^{t+1} \cup Y_{\mathbf{y}_j^t}^t$ dominate a $\mathbf{y}_i^t$, then it is only solutions at the (non-elite) end of domination chains in this subset which will be selected as the corresponding $\mathbf{y}_i^{t,*}$. This is because, for one solution to dominate another, it must be smaller (or equal) on each objective, and smaller on at least one. Therefore, for a chain to reach $\mathbf{y}_i^t$, then the elements closer to E^t, will also be *further* away in Euclidean objective space from $\mathbf{y}_i^t$ compared to other members of the chain.

By only selecting a member of E^t, the solutions in $Y_{\mathbf{y}_j^t}^t$ will only need to be compared to E^{t+1}, rather than to $E^{t+1} \cup Y_{\mathbf{y}_j^t}^t$.

By selecting a dominating member of $E^{t+1} \cup Y_{\mathbf{y}_j^t}^t$ at random, no distances need to be computed in objective space between the $E^{t+1} \cup Y_{\mathbf{y}_j^t}^t$ elements, also once the first dominating solution is identified, this may be used as $\mathbf{y}_i^*$ and the rest of the set need not be processed (however, to prevent the approach depending on the order of set union, the processing should to be randomised).

Note that when observing the $\mathbf{y}_i^{t,*}$ for each element of $Y^t \setminus E^t$ it will not necessarily be the closest dominating element in Y^t, as it may have been *chosen* at time $t-n$, and since then new closer dominating entrants may have entered Y, but not been compared against that particular element (as new solutions are compared only to E^t initially). Our single update approach does not guarantee a 'perfect' $\mathbf{y}_i^{t,*}$ will be maintained for each dominated $\mathbf{y}_i^t$, but it does ensure that the one assigned was the best at the time it was last updated (given the choice from $E^{t+1} \cup Y^t_{\mathbf{y}_j^t}$).

We now illustrate how this behaves by simulating the growth of a Y^t, whilst maintaining the links to single dominating members in order to determine E^t at each generation. We follow a general procedure where at each time step either the objective vector for an existing member of Y^t, is changed, or a new location, $\mathbf{y}^{t+1}_{new}$, is added. We alternate between each of these actions.

For each member of Y^t we store an underlying 'true' objective location, which is never observed directly. Instead, an evaluation of a design location results in a noisy version of the true objective vector (with additive Gaussian noise). The $\mathbf{y}$ of a solution is the mean of the objective vector resamples taken thus far for that member. We model the iterative generating process of Y^t in four distinct ways, based on two solution generation models, and two resampling models, which mimic both how a set may be maintained, and how an optimiser may progress over time.

For solution generation the two regimes are:

1. The underlying 'true' objective locations of new solutions are selected completely at random (drawn from a unit variance Gaussian). This emulates a very difficult search problem, where the objective vector of a new solution bear no relation to fitness of the best members evaluated thus far (the likely parents).

2. The underlying 'true' objective locations of new solutions are selected as a perturbed value in the proximity to the underlying (true) objective vector of a randomly selected member of E^t (perturbed via additive multivariate isotropic Gaussian noise with $\sigma = 0.2$). This simulates a search where the evolved solutions are in the general region of objective space as previously discovered good solutions, and emulates steady convergence over time.

For electing which $\mathbf{y}^t$ to vary at a time step:

1. The single $\mathbf{y}_j^t$ to change is selected at random from Y^t – *any* solution may have its objective vector changed.

2. The single $\mathbf{y}_j^t$ to change is selected at random from E^t – only estimated Pareto solutions have their objective vector changed.

We initialise Y^1 with a single location in objective space, whose true objective vector is $\mathbf{0}$, which is perturbed by additive multivariate isotropic Gaussian noise with $\sigma = 0.1$. We then run for a further 199 time steps, at each step either simulating the reevaluation of a location (with the noise on the true objective location having $\sigma = 0.1$), or by generating a new sample location (simulated a new solution entering Y^t). Figures 3-6 shows the

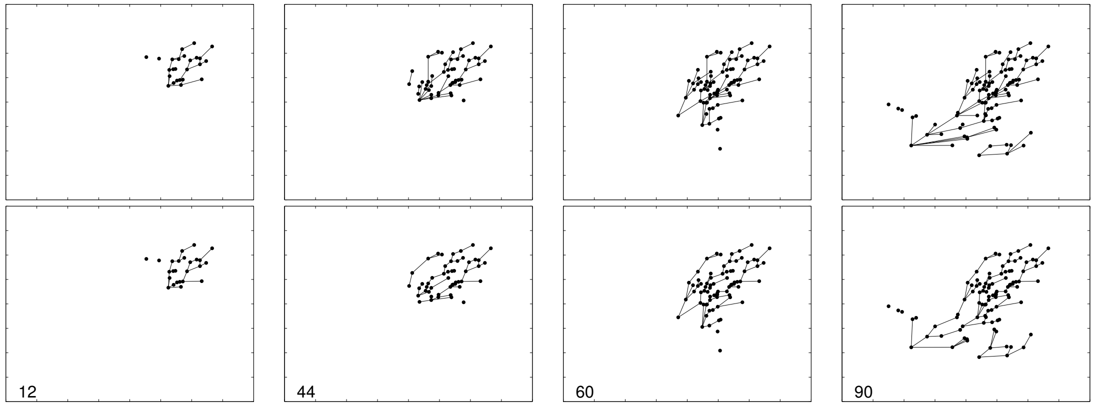

Figure 3: Resample only E^t members, converging algorithm simulation. *Top:* Maintaining single domination links, 200 simulated steps, $t = \{50, 100, 150, 200\}$ plotted left to right. *Bottom:* Corresponding optimal links if Y^t is considering in its entirety each time step when determining $\mathbf{y}_i^{t,*}$, rather than $X^t \cup Y^t_{\mathbf{y}^t_j}$ (for reevaluations) or E^t (for new locations). Number in lower left of panel indicates the Hamming distance between between the optimal and the maintained graph.

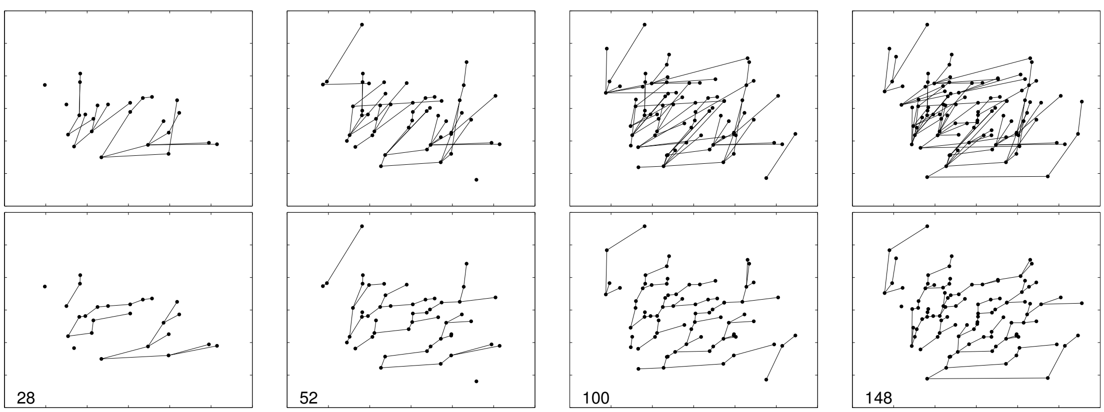

Figure 4: Resample only E^t members, random algorithm simulation. Panel arrangement as in Figure 3.

members of Y^t, and the maintained dominating member links, at $t = \{50, 100, 150, 200\}$ for each of the four regime combinations. The bottom row of panels on each of these the plots shows the population with 'perfect' domination links (i.e., each member of $Y^t \setminus E^t$ is connected to its closest dominating member in all of Y^t. The values in the lower left of each of these panels gives the Hamming distance between it and the panel directly above, which show the actual links that are maintained by the algorithm.

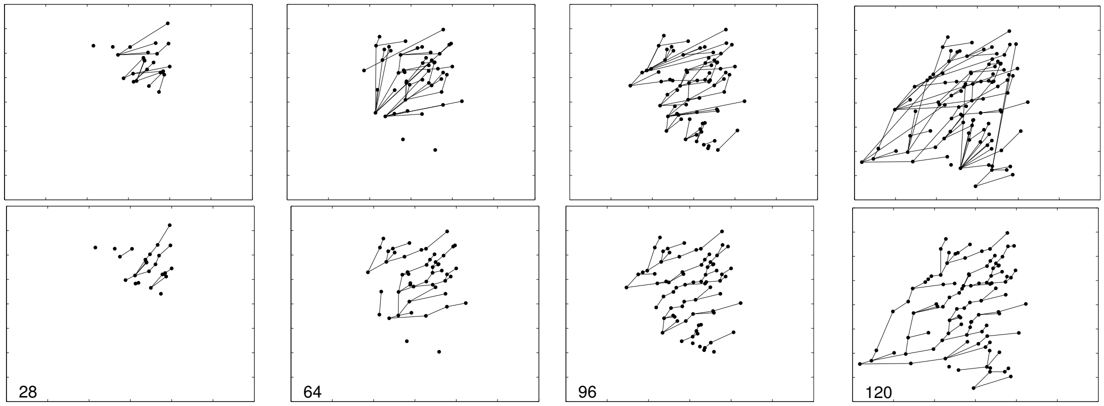

Figure 5: Resample only Y^t members, converging algorithm simulation. Panel arrangement as in Figure 3.

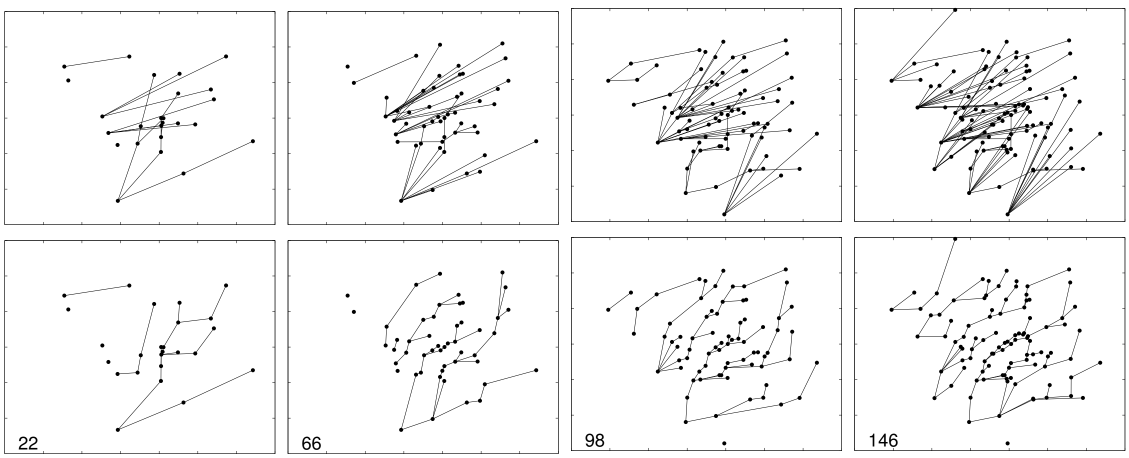

Figure 6: Resample only Y^t members, random algorithm simulation. Panel arrangement as in Figure 3.

All of the simulations vary from the optimal single edge allocation, with the random algorithm simulation, with Y^t selected from resampling at random, being the worst matching. The simulation with resamples from E^t exclusively, and converging algorithm, performs the best.

Figure 7 shows the mean set sizes under the four different regimes over 30 simulations for 20000 time steps, with the three single-link maintenance protocols. In the upper left of each panel is the average size of $|Y^t_{\mathbf{y}^t_j} \cup \{\mathbf{y}^{t+1}_j\} \setminus E^t|$ across the 30 runs and across each time step where there are resamples (and therefore where the members of $Y^t_{\mathbf{y}^t_j} \cup \{\mathbf{y}^{t+1}_j\} \setminus E^t$

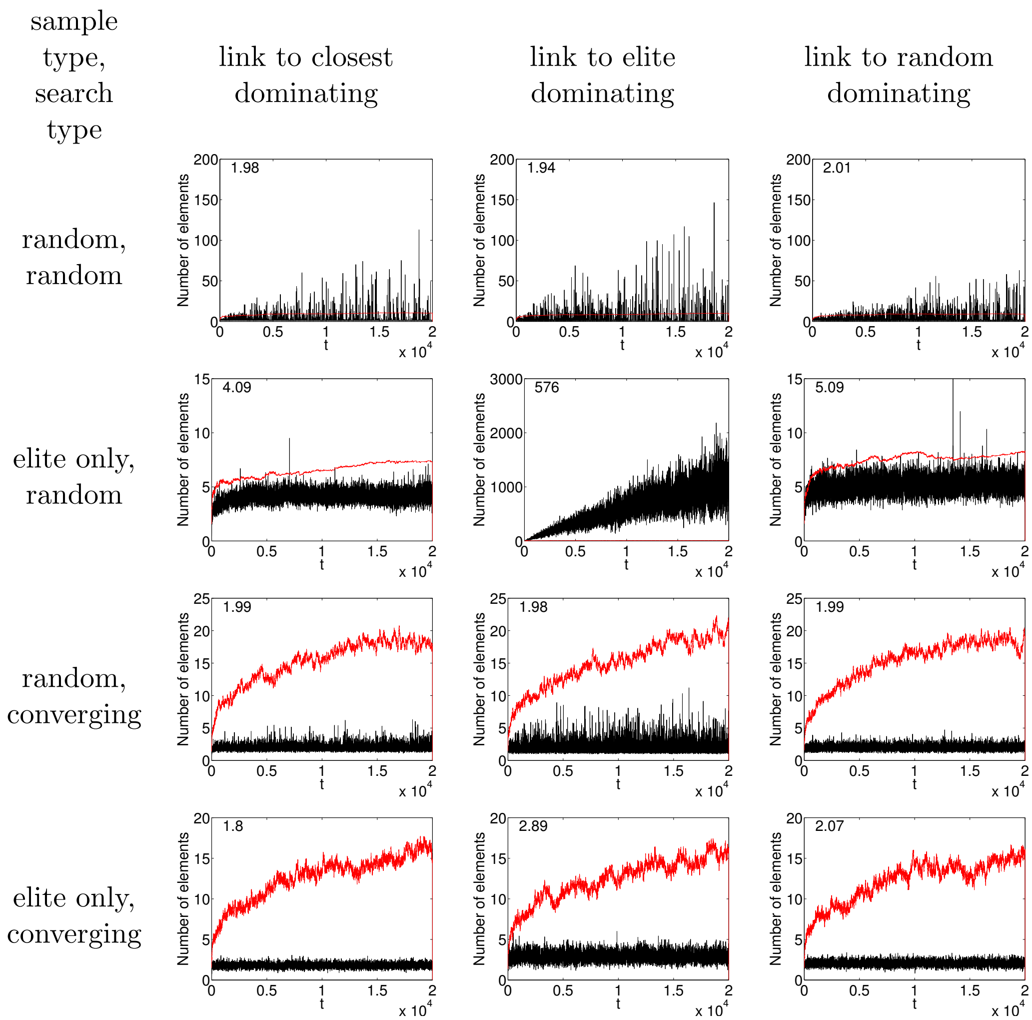

Figure 7: $|E^t|$ and $|Y^t_{\mathbf{y}^t_j} \cup \{\mathbf{y}^{t+1}_j\} \setminus E^t|$ under the four simulated regimes – plotted as red and black lines respectively – using three different link management regimes: closest dominating, elite member, and random dominating. Means of 30 simulations of each regime for 20000 time steps shown. Value in top left of panel gives average $|Y^t_{\mathbf{y}^t_j} \cup \{\mathbf{y}^{t+1}_j\} \setminus E^t|$ across all time steps where a $\mathbf{y}^t_j$ is resampled.

are compared to E^t, and each other). It can be seen that the computational complexity of updating the single links is extremely low in practice, apart from the situation where the search is random, and the resamples are selected at random from Y^t – however even in this situation the largest average $|Y^t_{\mathbf{y}^t_j} \cup \{\mathbf{y}^{t+1}_j\} \setminus E^t|$ at any particular times step is much lower

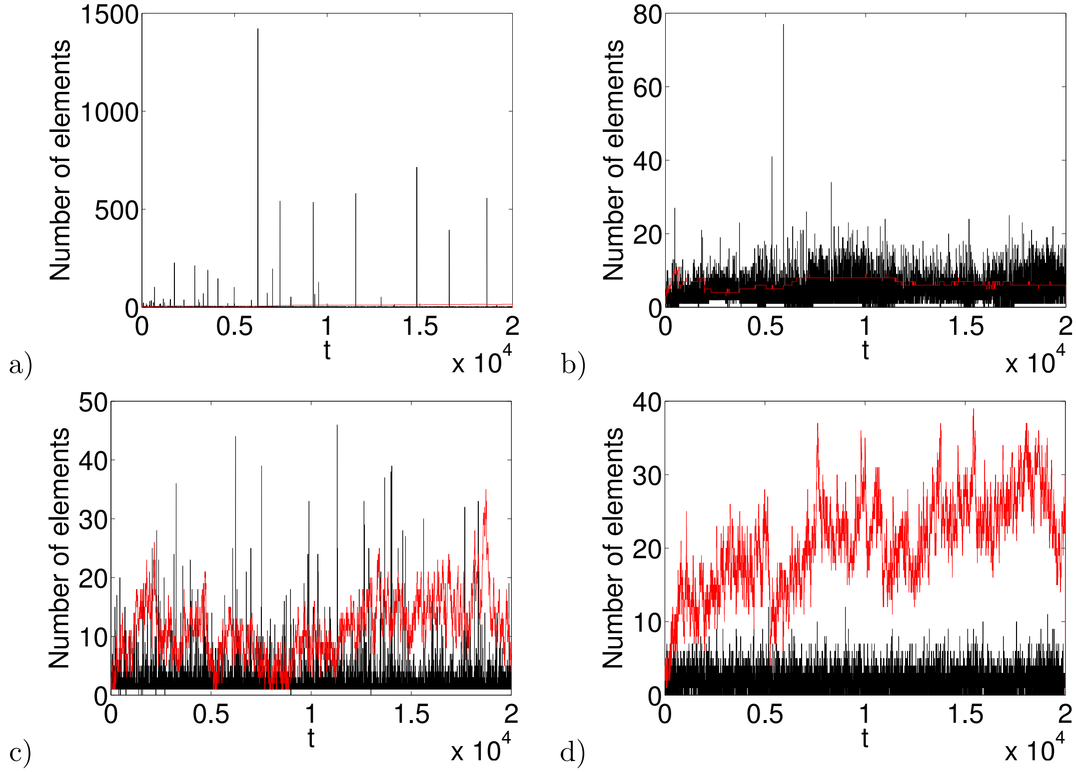

Figure 8: Plots of how $|E^t|$ and $|Y^t_{\mathbf{y}^t_j} \cup \{\mathbf{y}^{t+1}_j\} \setminus E^t|$ vary as time progresses under the four simulated regimes – plotted as red and black lines respectively for 20000 time steps. *(a)* Resample randomly in Y^t and random algorithm simulation. *(b)* Resample randomly in Y^t converging algorithm simulation. *(c)* Resample randomly in E^t and random algorithm simulation. *(d)* Resample randomly in E^t and converging algorithm simulation.

that the worst case possible of 9999 for all three link maintenance approaches. When the resampling is exclusively from E^t but the search behaves randomly, linking to the single closest dominating solution has the best performance. Selecting a dominating solution at random has slightly worse average performance, however selecting a dominating element only from E^t is extremely expensive – although in this situation members of $Y^t_{\mathbf{y}^t_j} \cup \{\mathbf{y}^{t+1}_j\} \setminus E^t$ are not required to have domination comparisons to each other to select their linked dominating solution (only to the members of E^t), the average size of $|Y^t_{\mathbf{y}^t_j} \cup \{\mathbf{y}^{t+1}_j\} \setminus E^t|$ is 576 (compared to only 4.1 and 5.1 for the other approaches) meaning the computational gains are rapidly exceeded by the costs of domination comparisons between $Y^t_{\mathbf{y}^t_j} \cup \{\mathbf{y}^{t+1}_j\} \setminus E^t$ and E^t.

Overall selecting the closest dominating linked solution is to be preferred based on the empirical assessment, as it constantly leads to equivalent or smaller $|Y^t_{\mathbf{y}^t_j} \cup \{\mathbf{y}^{t+1}_j\} \setminus E^t|$ the other two approaches across modelled regimes. Figure 8 shows the behaviour of this approach for a single run. The worst performing situation is when the resampled solutions are selected at random and the algorithm behaves like a random search (Figure 8a), with $|Y^t_{\mathbf{y}^t_j} \cup \{\mathbf{y}^{t+1}_j\} \setminus E^t|$ sporadically being orders of magnitude larger than $|E^t|$. This is because there can be long stretches of time when elements of E^t are not resampled (as they rapidly become a very small fraction of Y^t as time progress) or replaced (as random search means the probability of finding $\mathbf{y}^{t+1}_{new}$ which are non-dominated by E^t reduces as time progresses). This means that elements of E^t can accrue many links (through new locations having them selected as their $\mathbf{y}^{t,*}_i$), before a shift in the estimated location of a $\mathbf{e}^t \in E^t$ element due to a resample means all these linked locations need comparing to each other and the rest of E^t.

At the other end of the performance spectrum is the simulation where only E^t members are resampled and the algorithm has converging rather than completely random search properties (Figure 8d). In this case the regular resampling of E^t means that its members are prevented from accruing links to all but those in close (objective) proximity, and as E^t is also slowly moving forward the dominations chains tend to be much longer, meaning that if a member of Y^t enters E^t, due to a location shift of an element of E^t moving it from a dominating objective location, the new entrant to E^t will also tend to have only a few elements of Y^t linking to it. In this situation, the size of $|Y^t_{\mathbf{y}^t_j} \cup \{\mathbf{y}^{t+1}_j\} \setminus E^t|$ can be seen to be almost always smaller than E^t.

In the situation where only Y^t are resampled, but the new locations are simulating a converging search exploiting E^t (Figure 8c) the size of $|Y^t_{\mathbf{y}^t_j} \cup \{\mathbf{y}^{t+1}_j\} \setminus E^t|$ is sometimes above $|E^t|$, but only infrequently. In the simulation where only E^t members are resampled by the search is random, $|Y^t_{\mathbf{y}^t_j} \cup \{\mathbf{y}^{t+1}_j\} \setminus E^t|$ is regularly above $|E^t|$, however not excessively so, and $|E^t|$ itself tends to be smaller that the converging algorithms simulations.

Figure 9 shows the mean $|Y^t_{\mathbf{y}^t_j} \cup \{\mathbf{y}^{t+1}_j\} \setminus E^t|$ over 30 runs for 300000 function evaluations, on the CEC'09 test problems UP1-10 [7], using the RTEA noisy multi-objective optimiser [8], which resamples a solution every other function evaluation. The algorithm incorporated the closest dominating member approach for the maintenance of E^t, and as before the top right-hand value in the panel gives the average size of $|Y^t_{\mathbf{y}^t_j} \cup \{\mathbf{y}^{t+1}_j\} \setminus E^t|$ for resampling time steps of the algorithm ($\approx$ 150000) in total. For all problems and noise levels the average $|Y^t_{\mathbf{y}^t_j} \cup \{\mathbf{y}^{t+1}_j\} \setminus E^t|$ experienced is between 2 and 4.

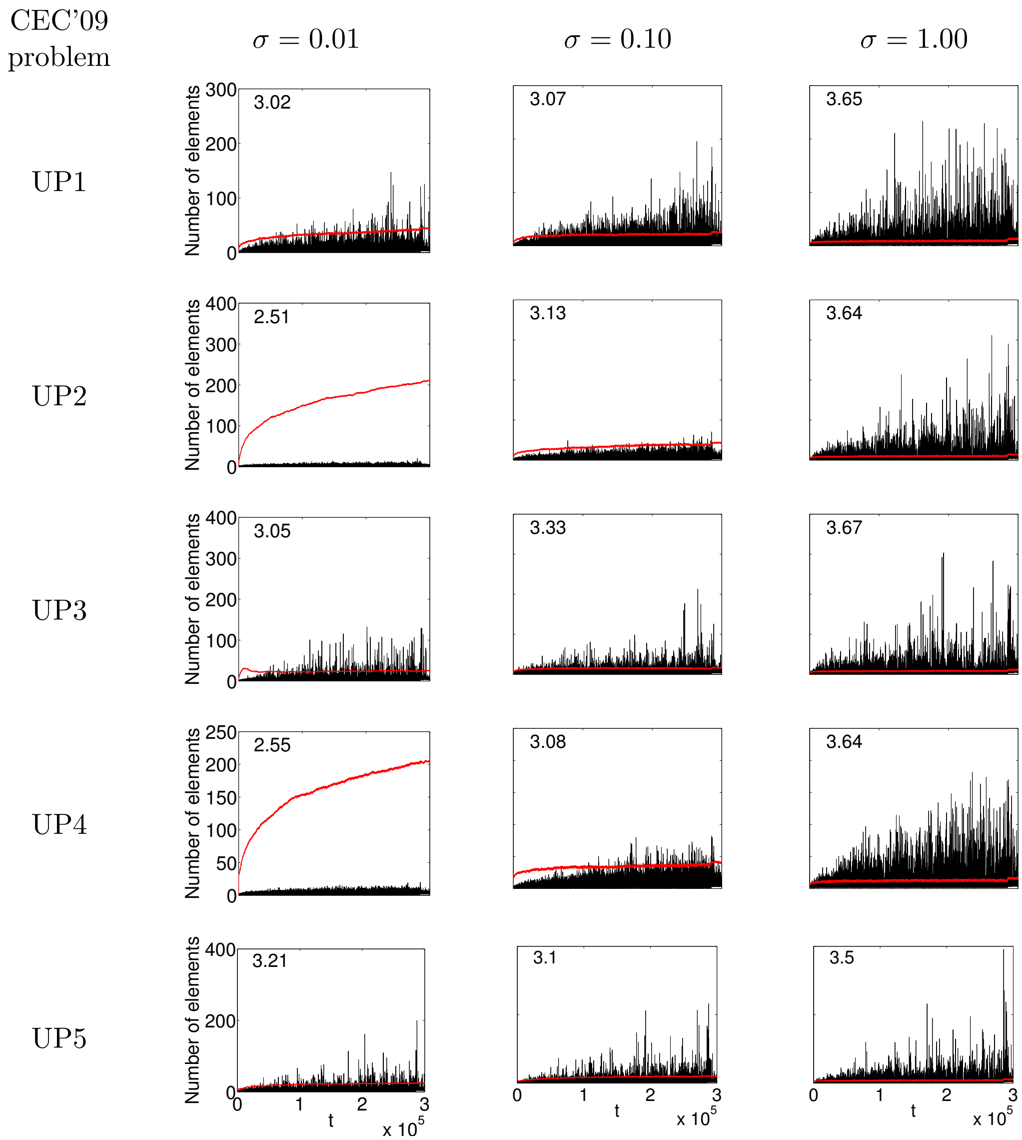

Figure 9: $|E^t|$ and $|Y^t_{\mathbf{y}^t_j} \cup \{\mathbf{y}^{t+1}_j\} \setminus E^t|$ – plotted as red and black lines respectively – using three closest dominating member link management regimes. Means of 30 simulations of each regime for 300000 function evaluations of the RTEA optimiser on the CEC'09 test problems 1-5. Value in top left of panel gives average $|Y^t_{\mathbf{y}^t_j} \cup \{\mathbf{y}^{t+1}_j\} \setminus E^t|$ across all time steps where a $\mathbf{y}^t_j$ is resampled.

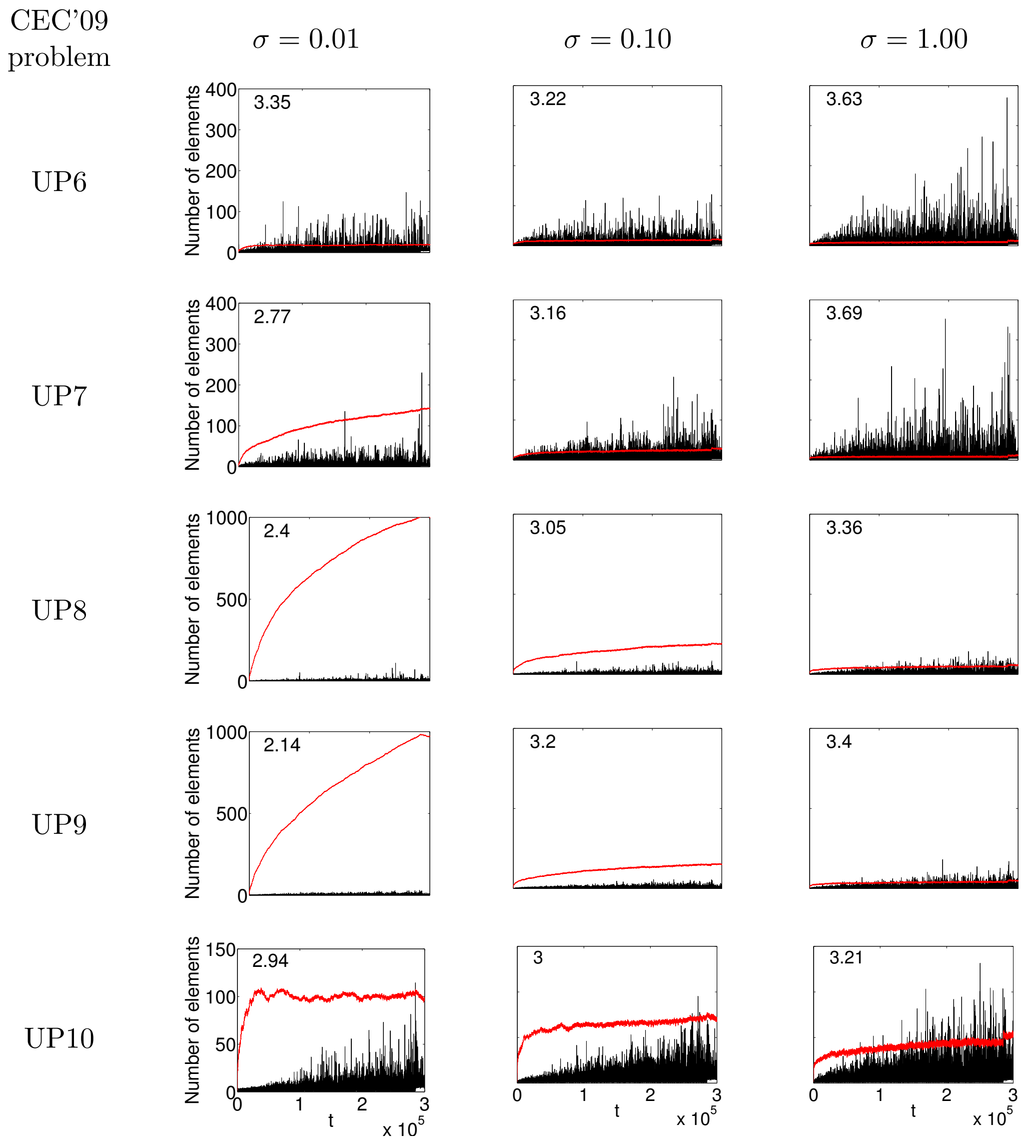

Figure 10: $|E^t|$ and $|Y^t_{\mathbf{y}^t_j} \cup \{\mathbf{y}^{t+1}_j\} \setminus E^t|$ – plotted as red and black lines respectively – using three closest dominating member link management regimes. Means of 30 simulations of each regime for 300000 function evaluations of the RTEA optimiser on the CEC'09 test problems 6-10. Value in top left of panel gives average $|Y^t_{\mathbf{y}^t_j} \cup \{\mathbf{y}^{t+1}_j\} \setminus E^t|$ across all time steps where an $\mathbf{y}^t_j$ is resampled.

6 Discussion

In this paper we have examined the situation where the assigned objective values for a solution may *change* at a later time point, and the implications for identifying and storing the best estimate of the Pareto set of *all* solutions evaluated thus far in an optimisation when this is the case. More generally, we are concerned with tracking the non-dominated subset of a general set of points Y, when Y may increase over time, and the members may have their values altered at a later date. By maintaining single tracked *dominators* for all members of Y at time t who are not non-dominated, incremental adjustments to the membership of Y^{t+1} (or additions to it) may be effectively managed. We have provided worst case complexities using this management routine, but show empirically (through both simulation and algorithm runs) that the effective complexity if this approach is, on average, much lower than the worst case complexity, and even when $|Y^t|$ has hundreds of thousands of elements, the number of domination comparisons required each time step are manageable.

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